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Functorial maximal spectra

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Abstract

Maximal spectra, of one kind or another, are usually not expected to be functorial, in notable contrast with the corresponding prime spectra, but there are cases in which the particular nature of the structures involved make them so. Here, we consider this in the context of *compact normal frames*, establishing the more fundamental functoriality of their saturation quotients which then readily implies that of their maximal spectrum, and showing this to be the common root of results concerning f-rings (J.R. Isbell, J. London Math. Soc. 40 (1965) 63), Gelfand rings (C.J. Mulvey, J. Algebra 56 (1979) 499), and the maximal ideals of $C(X)$ (I.M. Gelfand, A.N. Kolmogoroff, Doklady Acad. Nauk USSR 22 (1939) 11). © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

In various familiar situations, a certain space of "prime ideals" is functorially associated with given algebraic structures such as

> the usual prime (= Zariski) spectrum of a commutative ring with unit, the space of prime ideals of a lattice or a Boolean algebra, and the space of irreducible ℓ-ideals of a lattice-ordered ring.

In all such cases, the space involved contains the subspace of *maximal ideals* which might be considered of particular importance, but this tends not to give rise to a subfunctor, as is exemplified by the embedding of $\mathbb{Z}$ into $\mathbb{Q}$ or of the three-element chain into the four-element Boolean algebra. On the other hand, there are special

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situations in which the maximal spectra turn out to be functorial in their own right. Particular instances of this are the functoriality of the space of maximal ℓ-ideals of a commutative f-ring with unit [19] and of the space of maximal ideals of a Gelfand ring [24]. Further, the classical theorem that the identical embedding of $C^*(X)$ into $C(X)$ induces a homeomorphism between the corresponding maximal ideal spaces [14] may also be viewed as a case of this type.

The purpose of this paper is to exhibit a natural common root for these results and in the process to strengthen them by first establishing an appropriate pointfree version, that is, the functoriality of certain frames rather than spaces, from which that of the spaces involved then simply follows via the usual spectrum functor for frames. The proper setting for this will be *compact normal frames* and, specifically, their saturation quotients.

We begin with a brief survey of the background required, collecting several facts about frames on the one hand and ℓ-rings on the other (Section 1), and then proceed to establish (Section 2) the central result, the functoriality of the correspondence $L \mapsto SL$, the saturation quotient of L, for compact normal frames L, with the added result that, for any flat homomorphism $L \to M$ of such frames, the associated $SL \to SM$ is an isomorphism (2.2). This in turn will imply that $L \mapsto \operatorname{Max} L$, the space of maximal elements of L, is functorial here because $\operatorname{Max} L = \operatorname{Max}(SL) = \Sigma(SL)$ (2.4). In addition, we show that SL is isomorphic to the regular coreflection $\mathit{Reg}\, L$ of L (2.5) and derive the obvious application of (2.2) and (2.4) to bounded distributive lattices (2.6).

Next, we consider f-rings (Section 3) to which the results of Section 2 apply because the lattice $\mathfrak{L}A$ of ℓ-ideals of any commutative f-ring A with unit is a compact normal frame and functorial in A. Apart from the immediate conclusion that the correspondences $A \mapsto \mathfrak{M}A = S\mathfrak{L}A$ and $A \mapsto \operatorname{Max} A$, the space of maximal ℓ-ideals, are functorial (3.1) we derive from (2.2) that $\mathfrak{M}B \to \mathfrak{M}A$ is an isomorphism for the identical embedding $B \to A$ of any convex subring B of A which in particular applies to the bounded part A^* of A (3.3). This generalizes a result of [16] to arbitrary f-rings which in turn generalized the Gelfand–Kolmogoroff Theorem for the special case $A = C(X)$. In addition, we explain exactly how the latter is derived from the isomorphism (3.3), the important ingredient here being the role of *cozero sets* in relation to the saturated ℓ-ideals (3.4). Further, we discuss the corresponding result in the case of $A = C(X, \mathbb{Z})$ (3.5).

Finally, we turn to Gelfand rings (Section 4). Here, in order to create the setting to which the general results of Section 2 can be applied, we first establish the relevant properties of the lattice $\mathit{RId}\, A$ of *symmetric Baer radical ideals* of an arbitrary ring A with unit (4.2, with proofs given in the appendix), namely, that $\mathit{RId}\, A$ is a coherent frame, functorial in A (as is familiar in the commutative case but seems to be new in this generality). Next, we show, for *Gelfand rings*, that $\mathit{RId}\, A$ is normal (4.3) and identify $S(\mathit{RId}\, A)$ with the frame $\mathit{BId}\, A$ of Brown–McCoy radical ideals of A (4.4). This done, we have the obvious conclusion that the correspondences $A \mapsto \mathit{BId}\, A$ and $A \mapsto \operatorname{Max} A$, the space of maximal ideals of A, are functorial, where the second part is the result of [24] quoted earlier. Finally, we add some comments on the role of the

Axiom of Choice in the context of Gelfand rings, pointing out among other things that this holds iff $RId\,A$ is normal for every Gelfand ring.

To clarify the latter statement, it should be added that we take as our basis Zermelo–Fraenkel set theory (as usually understood, without the Axiom of Choice) unless explicitly stated otherwise in which case the corresponding result will also be marked by as asterisk.

1. Background

1.1. For general notions concerning frames and frame homomorphisms we refer to [20] or [26]. Here, we recall a few facts which will be of particular relevance in our context.

1.2. The frames of main interest here are the *compact normal* frames, meaning: the frames L such that

(1) for any $S \subseteq L$, $\bigvee S = e$ (the unit of L) implies $\bigvee T = e$ for some finite $T \subseteq S$, and
(2) if $a \vee b = e$ in L then there exist $c, d \in L$ such that $a \vee c = e = b \vee d$ and $c \wedge d = 0$.

KNFrm will be the corresponding full subcategory of the category **Frm** of all frames and frame homomorphisms.

1.3. As is familiar, the *spectrum* of a frame L may be described either as the space of all homomorphisms $\xi : L \to \mathbf{2}$ or as the space of all prime elements of L, that is, the $p \in L$ such that $p < e$ and $a \wedge b \leqslant p$ implies $a \leqslant p$ or $b \leqslant p$. As it turns out, the latter view will be more convenient here, and accordingly we let ΣL be the space of all prime elements of L, with the open sets

$$\Sigma_a = \{ p \in \Sigma L \mid a \nleqslant p \} \quad (a \in L).$$

Further, this determines a contravariant functor Σ from **Frm** to the category **Top** of topological spaces and continuous maps such that, for any frame homomorphism $h : L \to M$, $\Sigma h : \Sigma M \to \Sigma L$ takes $p \in \Sigma M$ to $h_*(p) \in \Sigma L$ where $h_* : M \to L$ is the *right adjoint* of h, characterized by the condition

$$h(a) \leqslant b \quad \text{iff} \quad a \leqslant h_*(b)$$

for all $a \in L$ and $b \in M$.

Note that h_* preserves arbitrary meets. Also, h is onto iff $hh_* = id_M$. Further, h is called *flat* if it is onto, $h_*(0) = 0$, and $h_*(a \vee b) = h_*(a) \vee h_*(b)$.

1.4. For any frame L, the *maximal* elements are obviously prime, and $\operatorname{Max} L$ will be the subspace of ΣL consisting of these elements. Note that the correspondence $L \mapsto \operatorname{Max} L$ does not define a subfunctor of Σ: simple examples show that there are homomorphisms $h : L \to M$ for which h_* does not take maximal to maximal elements—for any finite totally ordered L with more than two elements take $h : L \to L$ such that $h(a) = e$ for all $a > 0$.

1.5. On any frame L, we let $a \prec b$ (a is well inside, or rather below, b) signify that $b \vee a^* = e$ for the pseudo-complement $a^* = \bigvee\{x \in L \mid x \wedge a = 0\}$ of a. Similarly, we take $a \prec\!\!\prec b$ (a is really inside, or: completely below, b) to mean that there exists a sequence $(c_{nk})_{n=0,1,\ldots;\ k=0,1,\ldots,2^n}$ such that

$$c_{00} = a, \quad c_{01} = b, \quad c_{nk} = c_{n+1\ 2k}, \quad c_{nk} \prec c_{nk+1}$$

As is familiar, L is then called *regular* (*completely regular*) provided

$$a = \bigvee\{x \in L \mid x \prec a\} \quad \left(a = \bigvee\{x \in L \mid x \prec\!\!\prec a\}\right)$$

for all $a \in L$. Further, we note that any frame L contains a largest regular subframe $Reg\, L$ and the identical embedding $Reg\, L \to L$ is then the coreflection map to L from regular frames; also, the analogous result holds for complete regularity.

For homomorphisms between regular frames one has the following facts:
(1) Any codense $h : L \to M$ ($h(a) = e$ implies $a = e$) is one–one.
(2) Any dense $h : L \to M$ ($h(a) = 0$ implies $a = 0$, or $h_*(0) = 0$) is monic.
(3) If $f, g : L \to M$ are such that $f \leqslant g$, that is, $f(a) \leqslant g(a)$ for all $a \in L$, then $f = g$.
Further, $\Sigma L = \mathrm{Max}\, L$ for any regular frame.

1.6. An element c of a frame L is called *compact* (or finite) if $c \leqslant \bigvee S$ implies $c \leqslant \bigvee T$ for some finite $T \subseteq S$. L is called *coherent* provided it is compact, the meet of any two compact elements is compact, and every element is a join of compact elements. Finally, a homomorphism between coherent frames is called coherent if it takes compact elements to compact elements.

1.7. Any compact frame L has a particular nucleus, its *saturation nucleus* s_L defined by

$$s_L(a) = \bigvee\{x \in L \mid x \vee y = e \text{ implies } a \vee y = e \text{ for all } y \in L\}.$$

The $x \in L$ which occur here are called a-small, and $s_L(a)$ is the largest a-small element in L.

The corresponding frame $Fix\, s_L$ is called the *saturation quotient* SL of L. It is compact since s_L is codense. Further, $SL = L$ iff L is subfit, that is, for any $a < b$ in L there exist $c \in L$ such that $a \vee c < e = b \vee c$, and $s_L : L \to SL$ is the unique codense homomorphism of L onto subfit frames [7].

1.8. Recall that a *lattice ordered ring* (ℓ-ring) is a ring A with a lattice structure such that $A^+ = \{a \in A \mid a \geqslant 0\}$ is closed under addition and multiplication. An ℓ-ring is called an f-ring if $(a \wedge b)c = (ac) \wedge (bc)$ for all $a, b \in A$ and $c \in A^+$. We shall primarily be concerned with commutative f-rings with unit and their ℓ-ring homomorphisms; **FAnn** will then be the resulting category. As general references we suggest [12,20]. For the particular points mentioned below, see also [5].

1.9. An ℓ-ideal in an ℓ-ring A is a ring ideal J of A for which $|a| \leqslant |b|$ and $b \in J$ implies $a \in J$. The lattice $\mathfrak{L}A$ of all ℓ-ideals of A is a compact frame, and join in $\mathfrak{L}A$ is the same as in the lattice of all ring ideals of A. Further, the correspondence $A \mapsto \mathfrak{L}A$ is functorial such that, for any ℓ-ring homomorphism $\varphi : A \to B$, $\mathfrak{L}\varphi : \mathfrak{L}A \to \mathfrak{L}B$ takes $J \in \mathfrak{L}A$ to the ℓ-ideal of B generated by its image $\varphi[J]$.

For the principal ℓ-ideals

$$[a] = \{x \in A \mid |x| \leqslant |a|b \text{ for some } b \in A^+\}$$

in any ℓ-ring A, $[a] + [b] = [|a| \vee |b|]$ so that any finitely generated ideal is actually principal. On the other hand, $[a] \wedge [b] = [a \wedge b]$ for all $a, b \in A^+$ iff A is an f-ring and consequently $\mathfrak{L}A$ is coherent for any such A. Moreover, $\mathfrak{L}A$ is normal for any f-ring A.

1.10. An f-ring is said to *have bounded inversion* (or called *strong*) if every $s \geqslant 1$ in A is invertible in A. Any f-ring A has a *strong envelope* $\tilde{A} \supseteq A$ which is the unique strong f-ring that contains A as ℓ-subring and is generated over A by the inverses of the $s \geqslant 1$ in A. Moreover, the functor $\mathfrak{L}$ turns the identical embedding $A \to \tilde{A}$ into an isomorphism $\mathfrak{L}A \to \mathfrak{L}\tilde{A}$.

2. Compact normal frames

2.1. We recall a few basic results concerning compact normal frames L from [5].

2.1.1. The saturation quotient SL is compact regular.

2.1.2. The homomorphism $s_L : L \to SL$ has a right inverse $r_L : SL \to L$ such that $r_L(a) = \bigvee\{x \in L \mid x \prec a \text{ in } L\}$.

2.1.3. $\Sigma(SL) = \mathrm{Max}(SL) = \mathrm{Max}\, L$.

2.2. Proposition. *The correspondence $L \mapsto SL$ is functorial on* **KNFrm** *such that*

$$Sh(a) = s_M \left(\bigvee\{h(x) \mid x \prec a \text{ in } L\} \right) = s_M h r_L(a)$$

for any $h : L \to M$. Moreover, Sh is an isomorphism whenever h is flat, with inverse $s_L h_* \mid SM$.

Proof. Since Sh is clearly a homomorphism we only have to check the functoriality of the correspondence $h \mapsto Sh$. For $h = id_L$, the desired result is obvious by 2.1.2. On the other hand, for any $h : L \to M$ and $g : M \to N$,

$$(Sg)(Sh) = s_N g r_M s_M h r_L \leqslant s_N g h r_L = S(gh)$$

because

$$r_M s_M(a) = \bigvee\{x \in M \mid x \prec s_M(a) \text{ in } M\} = r_M(a) \leqslant a,$$

the second step since $x \prec s_M(a)$ iff $x \prec a$ by the definition of $\prec$ and s_M. It follows that $(Sg)(Sh) = S(gh)$ by 1.5.

For the second part of the proposition, let $h: L \to M$ now be flat (1.3). To see that Sh is an isomorphism we show it is codense and onto.

If $Sh(a) = e$ then also $hr_L(a) = e$ since s_M is codense and hence $h(x) = e$ for some $x \prec a$ in L by compactness. It follows that $h(x^*) = 0$, hence $x^* = 0$ since h is dense, and consequently $a = e$ because $a \vee x^* = e$. Thus, Sh is codense. Further, for any $a \in SM$, put $b = s_L(h_*(a))$. Then

$$Sh(b) = s_M hr_L s_L h_*(a) = s_M hr_L h_*(a) \leqslant a$$

since $r_L s_L = r_L$ as shown above, $hh_* = id_M$ because h is onto, and $a \in SM$. For the reverse inequality we show that a is c-small where $c = hr_L(b)$. If $a \vee y = e$ in M then $h_*(a) \vee h_*(y) = e$ in L by the hypothesis on h, and by normality there exist disjoint $u, v \in L$ for which

$$h_*(a) \vee u = e = h_*(y) \vee v.$$

It follows that $v \prec h_*(a)$, hence $v \prec b$, and therefore $h(v) \leqslant c$; on the other hand, $h(v) \vee y = e$ so that $c \vee y = e$, the desired conclusion.

Finally, it is obvious from the definition of b that $(Sh)^{-1} = s_L h_* \mid M$. □

Remark 1. The assumption of normality is necessary for the first part of the proposition: if L is compact such that the map $Sh = s_M hr_L$ is a frame homomorphism for any homomorphism $h: L \to M$ with compact M then L is normal. If $a \vee b = e$ in L then also $s_L(a) \vee s_L(b) = e$ in SL, and using the case $M = L$ and $h = id_L$ it follows that $s_L r_L s_L(a) \vee s_L r_L s_L(b) = e$ in SL, hence $r_L s_L(a) \vee r_L s_L(b) = e$ in L since s_L is dense, and finally $r_L(a) \vee r_L(b) = e$ in L because $r_L s_L = r_L$. Now, by compactness, this implies $x \vee y = e$ for some $x \prec a$ and $y \prec b$, and consequently $a \vee x^* = e = b \vee y^*$ while $x^* \wedge y^* = 0$.

Remark 2. The correspondence $L \mapsto SL$ is not functorial for compact frames in general, in fact, not even for arbitrary *finite* frames [9].

2.3. The above formula for Sh is simple enough but there are some special cases in which it can be considerably simplified in that the reference to the saturation nucleus of M can be removed. In the following, KM stands for the set of compact elements of M.

Supplement. (1) *If $h: L \to M$ is flat and M is coherent then*

$$Sh(a) = \bigvee \{c \in KM \mid h_*(c) \leqslant a\}.$$

(2) *If L, M, and $h: L \to M$ are all coherent and h is flat then h_* is a frame homomorphism and $Sh = h_{**} \mid SL$.*

Proof. (1) Since $Sh(a)$ is the join of all compact $c \leqslant Sh(a)$ in M it will be enough to show that

$$c \leqslant Sh(a) \quad \text{iff} \quad h_*(c) \leqslant a$$

for any $a \in SL$ and $c \in KM$.

($\Rightarrow$) By Proposition 2.2, $s_L h_* Sh(a) = a$, hence $h_* Sh(a) \leqslant a$, and $c \leqslant Sh(a)$ clearly implies $h_*(c) \leqslant a$.

($\Leftarrow$) We first show that $h_* s_M(c)$ is $h_*(c)$-small. If $h_* s_M(c) \vee y = e$ in L then also $h_* s_M(c) \vee z = e$ for some $z \prec y$ by normality and hence $c \vee h(z) = e$ by applying h and using the definition of s_M; further, $h_*(c) \vee h_* h(z) = e$ by the flatness of h, and since $z \prec y$ and h is dense we conclude that $h_*(c) \vee y = e$, as claimed.

Now let $h_*(c) \leqslant a$. Then $s_L h_*(c) \leqslant a$, hence $s_L h_* s_M(c) \leqslant a$ by what was just shown, and by applying Sh we obtain $s_M(c) \leqslant Sh(a)$ by Proposition 2.2 and hence $c \leqslant Sh(a)$.

(2) The desired result will follow trivially from the formula in (1) once one knows that h_* is in fact a frame homomorphism. To see this it is sufficient to check that $h_*(\bigvee S) \leqslant \bigvee h_*[S]$ for any updirected $S \subseteq M$, and by coherence it is enough to show that $c \leqslant h_* \ (\bigvee S)$ implies $c \leqslant \bigvee h_*[S]$ for any compact $c \in L$–but this is any easy consequence of the fact that h is coherent. $\square$

2.4. Proposition. *The correspondence* $L \mapsto \operatorname{Max} L$ is functorial on **KNFrm** *such that* $\operatorname{Max} h = s_L h_* \mid \operatorname{Max} M$ *for any* $h : L \to M$.

Proof. The functoriality as such being obvious by 2.2 and 2.1.3, it only has to be checked that the continuous map $\operatorname{Max} M \to \operatorname{Max} L$ resulting from $h : L \to M$ by the functor ΣS does have the form indicated. Now, this map is effected by $(Sh)_*$ (1.3), and for any $u \in \operatorname{Max} L$ and $v \in \operatorname{Max} M$ we have

$$Sh(u) \leqslant v \quad \text{iff} \quad s_M\ h\ r_L(u) \leqslant v \quad \text{iff} \quad hr_L(u) \leqslant v$$

$$\text{iff} \quad r_L(u) \leqslant h_*(v) \quad \text{iff} \quad u \leqslant s_L h_*(v)$$

the final step since $s_L r_L = id_L$ (2.1.2) and $r_L s_L = r_L$ by the proof of Proposition 2.2. $\square$

Remark 1. The following elucidates the above description of $\operatorname{Max} h$. For any $h : L \to M$, the associated $\Sigma h : \Sigma M \to \Sigma L$ is effected by h_* (1.3) but h_* need not take maximal to maximal elements. On the other hand, for any prime p in a compact normal frame L, $s_L(p)$ is maximal and hence the only maximal element above p ([5], 1.4). Thus $\operatorname{Max} h$: $\operatorname{Max} M \to \operatorname{Max} L$ takes $u \in \operatorname{Max} M$ to the *unique maximal element of* L *above the prime element* $h_*(u)$. It should be added that the present proposition could be obtained easily enough without Proposition 2.2 by showing directly that the map $\Sigma L \to \operatorname{Max} L$ given by s_L is continuous for any compact normal frame L. Furthermore, Proposition 2.2 would actually follow from this provided the Prime Ideal Theorem (PIT) for Boolean algebras is assumed since that makes all compact regular frames spatial. Of course, the point here is that this proposition holds without any choice assumptions and hence constitutes the primary result in this context.

Remark 2. Given the Axiom of Choice, a *coherent* frame L is normal iff

(UM) *Every prime element of* L *is below a unique maximal element of* L,

as follows from [13] together with Hochster's Theorem that every spectral space is the prime spectrum of a commutative ring with unit [17]; for an elegant direct proof of this see [20]. Actually, a slight modification of the latter shows that PIT is already sufficient for ($\Leftarrow$) while ($\Rightarrow$) implies PIT because if it fails there exist non-normal coherent frames without any prime elements which then satisfy UM *vacuously*. We omit the details. Other than that, it should be pointed out that normality is not characterized by UM for compact frames in general: there are compact frames L such that every prime element of L is actually maximal which are not normal, as any non-Hausdorff compact sober T_1-space shows. A particularly simple example of such a space is the compactification of any infinite discrete space X by adjoining *two distinct* points a and b with respective basic neighbourhoods $U \cup \{a\}$ and $U \cup \{b\}$, $U \subseteq X$ cofinite.

2.5. In the following, we provide an alternative view of the saturation quotient SL of a compact normal frame L.

Recall that any frame L has a largest regular subframe $Reg\,L$ for which the identical embedding $Reg\,L \to L$ is then the coreflection map from regular frames (1.5). Further, for any homomorphism $h: L \to M$, $Reg\,h: Reg\,L \to Reg\,M$ is given by the action of h. Now, if L is compact normal then $s_L: L \to SL$ maps $Reg\,L$ one–one because it is codense (1.5). On the other hand, we have the homomorphism $r_L: SL \to L$, right inverse to s_L (2.1.2), and since $Im(r_L) \subseteq Reg\,L$ it follows that s_L maps $Reg\,L$ onto SL. Hence the result:

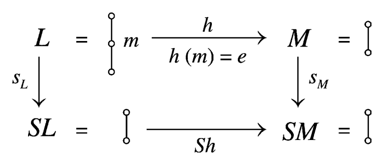
For any compact normal frame L, s_L *induces an isomorphism* $Reg\,L \to SL$ *with inverse given by* r_L.

Moreover, one can obviously derive Proposition 2.2 from this observation: any homomorphism $h: L \to M$ of compact normal frames induces the composite homomorphism

$$SL \xrightarrow[r_L]{\sim} Reg\,L \xrightarrow[Reg\,h]{} Reg\,M \xrightarrow[s_M]{\sim} SM,$$

functorial in h because $Reg\,h$ is, and this is exactly the Sh of Proposition 2.2.

One might add that, although the identical embeddings $Reg\,L \to L$ clearly provide a natural transformation from the functor Reg to the identity functor, their counterparts $s_L: L \to SL$ are *not* natural in L: in the following diagram



$s_M h \neq (Sh)s_L$ since $s_L(m) = 0$ but $s_M h(m) = e$.

2.6. Next we present an immediate application of 2.2 and 2.4 to the category $\mathbf{D}$ of bounded distributive lattices and their homomorphisms.

Let $\mathfrak{J}:\mathbf{D}\to\mathbf{Frm}$ be the functor taking each $A\in\mathbf{D}$ to its ideal lattice $\mathfrak{J}A$ and each $\varphi:A\to B$ in $\mathbf{D}$ to the homomorphism

$$\mathfrak{J}\varphi:\mathfrak{J}A\to\mathfrak{J}B,\ J\mapsto\bigcup\{\downarrow\varphi(a)\mid a\in J\}.$$

Now, $\mathfrak{J}A$ is compact (actually, of course, coherent), and the familiar, and obvious, fact that $\mathfrak{J}A$ is normal iff A is normal therefore implies the following

Proposition. *For the normal $A\in\mathbf{D}$, the correspondences $A\mapsto S\mathfrak{J}A$ and $A\mapsto \mathrm{Max}\,(\mathfrak{J}A)$ are functorial such that*

$$S\mathfrak{J}\varphi(J)=s_{\mathfrak{J}B}\left(\bigcup\{\downarrow\varphi(a)\mid a\prec b \text{ for some } b\in J\}\right)$$

and

$$\mathrm{Max}\,\mathfrak{J}\varphi(P)=s_{\mathfrak{J}A}(\varphi^{-1}[P]).$$

Proof. The first formula easily follows from Proposition 2.2 by taking the special features of $L=\mathfrak{J}A$ into account while the second results from Proposition 2.4 by the simple observation that $(\mathfrak{J}\varphi)_*(J)=\varphi^{-1}[J]$. □

Remark. $A\mapsto\mathrm{Max}\,(\mathfrak{J}A)$ is not functorial on all of $\mathbf{D}$, in fact not even on the subcategory of all finite $A\in\mathbf{D}$, as shown by Banaschewski and Vermeulen [9].

*__2.7.__ We conclude with an observation concerning the normality of compact frames in the presence of the Axiom of Choice. Given this, *a compact frame L is normal iff, for any distinct maximal elements $u,v\in L$, there exist $a\nleq u$ and $b\nleq v$ such that* $a\wedge b=0$.

For the non-trivial part, note that the second condition holds iff, for any distinct maximal $u,v\in L$, there exist $b\in L$ such that

$$b\prec u \quad\text{and}\quad b\nleq v$$

and this is the same as saying that u is the only maximal element above $r_L(u)=\bigvee\{x\in L\mid x\prec u\}$, for any maximal $u\in L$. Further, the latter implies that $r_L(a)\nleq u$ whenever $a\nleq u$ for any $a\in L$ and maximal $u\in L: a\vee r_L(u)=e$ by Zorn's Lemma since there is no maximal element above $a\vee r_L(u)$, hence $a\vee x=e$ for some $x\prec u$ by compactness, and therefore $x^*\prec a$ while $x^*\nleq u$, which proves the claim.

To obtain normality from this, let $a\vee b=e$ in L. Then, for any maximal $u\in L$, $a\nleq u$ or $b\nleq u$, hence also $r_L(a)\nleq u$ or $r_L(b)\nleq u$ showing that $r_L(a)\vee r_L(b)=e$, again by Zorn's Lemma. It follows that $x\vee y=e$ for some $x\prec a$ and $y\prec b$ by compactness, and consequently $a\vee x^*=e=b\vee y^*$ while $x^*\wedge y^*=0$.

It should be noted that the use of the Axiom of Choice is unavoidable here: whenever it fails one can obtain a compact (in fact: coherent) frame L without maximal elements which is not normal, using the corresponding partially ordered set of all partial choice functions. This L then satisfies the above weaker condition *vacuously* and hence shows that ($\Leftarrow$) does not hold.

We note that for *coherent* frames, the above equivalence is contained in V, 3.7 of [20] but the proof there specifically depends on coherence.

3. f-Rings

3.1. As noted in 1.9, the lattice $\mathfrak{L}A$ of all ℓ-ideals of an f-ring A (always assumed to be commutative with unit 1) is coherent and normal, providing a functor $\mathfrak{L}:\mathbf{FAnn}\to\mathbf{KNFrm}$. Consequently, putting $\mathfrak{M}A=S\mathfrak{L}A$ for the frame of saturated ℓ-ideals of A and $\operatorname{Max}A=\Sigma\mathfrak{M}A$ for the space of maximal ℓ-ideals of A, 2.2 and 2.4 yield the following.

Proposition. *The correspondences $A\mapsto\mathfrak{M}A$ and $A\mapsto\operatorname{Max}A$ are functorial such that*

$$\mathfrak{M}\varphi(J)=s_{\mathfrak{L}B}\left(\bigcup\{[\varphi(a)]\mid |a|\wedge|1-b|=0 \text{ for some } b\in J\}\right)$$

and

$$\operatorname{Max}\varphi(P)=s_{\mathfrak{L}A}\ (\varphi^{-1}[P])$$

for any $\varphi:A\to B$.

Proof. Regarding the first formula, it is clear that the $I\prec J$ in $\mathfrak{L}A$ in the version immediately derived from Proposition 2.2 can be replaced by the principal ℓ-ideals $[a]\prec J$, and an easy calculation then shows that the $a\in A$ which satisfy this are exactly the $a\in A$ specified above. The second formula results in the same way as the corresponding one in Proposition 2.6. $\square$

Remark 1. Beyond the fact that the frame $\mathfrak{M}A$ is compact regular for any f-ring A by 1.9 and 2.1.1, the special features of f-rings imply further that $\mathfrak{M}A$ is actually *completely regular*. To see this it is enough to show that $I\prec\!\!\prec J$ whenever $I\prec J$ in $\mathfrak{M}A$, and by (1.10) we may assume for the sake of simplicity that A has bounded inversion. Now, it readily follows from $I\prec J$ in $\mathfrak{M}A$ that there exist $c,d\in A^+$ such that

$$1=c\vee d,\ c\in J,\quad\text{and}\quad Id\subseteq\langle 0\rangle,$$

where $\langle a\rangle$ is the saturation of the principal ℓ-ideal $[a]$, and simple calculations then show that I is $[(c-p)^+]$-small and hence $I\subseteq\langle(c-p)^+\rangle$ for all $0<p<1$. Further, $\langle(c-p)^+\rangle\prec\!\!\prec\langle c\rangle$ by the proof of Proposition 2.3 in [5], and this implies that $I\prec\!\!\prec J$.

Remark 2. For any f-ring with bounded inversion, $\mathfrak{M}A=SId\,A$, the latter defined by the saturation operator considered on the complete lattice $Id\,A$ of *all* ideals of A [8], and consequently $\operatorname{Max}A$ is then just the space of maximal ring ideals of A.

Remark 3. The functoriality of $A \mapsto \operatorname{Max} A$ will obviously be of particular interest if (1) all $\operatorname{Max} A \neq \emptyset$ and (2) all $\operatorname{Max} A$ are compact. It turns out that either of these conditions alone is equivalent to the PIT. For (1) this is fairly obvious, the slightly less immediate ($\Rightarrow$) resulting from considering, for any Boolean algebra B, the Boolean power of the f-ring $\mathbb{Q}$ relative to B as in [1]. Regarding (2), ($\Leftarrow$) is again clear because here all $\mathfrak{M}A$ are spatial. For the more surprising ($\Rightarrow$) one notes that (2) makes the familiar embedding

$$X \to \operatorname{Max}(C^*(X)), \quad x \mapsto \{a \in C^*(X) \mid a(x) = 0\}$$

a compactification for each Tychonoff space X, and since $\operatorname{Max}(C^*(X))$ depends functorially on X (for instance: by the proposition) it provides the reflection to compact Tychonoff spaces. This in turn shows, by general principles, that products of such spaces are compact which is known to imply the PIT [22].

3.2. The second part of Proposition 3.1 is due to Isbell [19], as noted earlier. It may be instructive to compare the original proof of this with the one presented here. In [19], the starting point is the complete Boolean algebra of all $J \in \mathfrak{L}A$ such that $J = J^{**}$ in which one considers the ideals $\mathfrak{C}$ such that, for each $J \in \mathfrak{C}$ there exist $I \in \mathfrak{C}$ for which $J \prec\!\!\prec I$ in $\mathfrak{L}A$. Expressed in the general context of compact normal frames L, this amounts to looking at the ideals H of the complete Boolean algebra $\mathfrak{B}L = \{a \in L \mid a = a^{**}\}$ which are *completely regular* in L, that is, for each $a \in H$ there exist $b \in H$ such that $a \prec\!\!\prec b$ in L. Now, it is clear that the intervention of $\mathfrak{B}L$ is not really needed here: the lattice of these particular ideals of $\mathfrak{B}L$ is isomorphic to the lattice $CR\mathfrak{J}L$ of completely regular ideals of L itself because $a \prec\!\!\prec b$ in L implies $a^{**} \prec\!\!\prec b$. On the other hand, $CR\mathfrak{J}L$ one recognizes—with the wisdom of hindsight—as a familiar object, namely the compact completely regular coreflection βL of L [8]. Expressed in this language, [19] obtained the desired result by proving that (1) $A \mapsto \Sigma(\beta(\mathfrak{L}A))$ is functorial, and (2) $\operatorname{Max} A \cong \Sigma(\beta(\mathfrak{L}A))$. It follows that this approach amounts to exactly the same as ours, given Remark 1 of 3.1 and the obvious observation that $SL \cong Reg\,L \cong \beta L$ for any compact normal L with completely regular SL.

3.3. We now turn to an application of 2.3.

An ℓ-subring B of an ℓ-ring A is called *convex* (in A) if $0 \leqslant a \leqslant b$ for $a \in A$ and $b \in B$ implies $a \in B$.

The most natural example of this is the *bounded part* A^* of A, consisting of all $b \in A$ such that $|b| \leqslant n$ for some natural n. This is obviously an ℓ-subring of A by the familiar rules about $|\cdot|$ and trivially convex. Moreover, it is the smallest convex subring of A.

Below we shall make use of the following simple rules regarding $+$ and $\vee$:

$$x + y = (x \vee y) + (x \wedge y) = (x + (x \wedge y)) \vee (y + (x \wedge y)).$$

Now we have

Proposition. *For any convex subring B of an f-ring A, the functor $\mathfrak{M}$ turns the identical embedding $\varepsilon: B \to A$ into an isomorphism $\mathfrak{M}B \to \mathfrak{M}A$ taking J to $\{a \in A \mid [a] \cap B \subseteq J\}$.*

Proof. We first show that $\mathfrak{L}\varepsilon$ is flat. It is obviously dense and $(\mathfrak{L}\varepsilon)_*(I) = \varepsilon^{-1}[I] = I \cap B$ for any $I \in \mathfrak{L}A$. Next, $\mathfrak{L}\varepsilon$ is onto because each $[a], a \in A^+$, is generated by $a \wedge 1$ since

$$[a] = [a] \cap [1] = [a \wedge 1]$$

and $a \wedge 1 \in B$. Finally, if $a \in (I + J) \cap B$ for any $I, J \in \mathfrak{L}A$ then $a = b + c$, for some $b \in I$ and $c \in J$, and consequently

$$|a| \leqslant |b| + |c| = (|b| + (|b| \wedge |c|)) \vee (|c| + (|b| \wedge |c|))$$

by the rule quoted above. Here $\bar{b} = |b| + (|b| \wedge |c|) \in I$ and similarly $\bar{c} \in J$ so that

$$|a| = (|a| \wedge \bar{b}) \vee (|a| \wedge \bar{c}) \in I \cap B + J \cap B,$$

showing $(I + J) \cap B \subseteq (I \cap B) + (J \cap B)$, the non-trivial part of the desired identity.

It now follows from Proposition 2.2 that $\mathfrak{M}\varepsilon$ is an isomorphism, and since $\mathfrak{L}B, \mathfrak{L}A$, and $\mathfrak{L}\varepsilon$ are all coherent 2.3 says that the image of any $J \in \mathfrak{M}B$ by $\mathfrak{M}\varepsilon$ is the largest of the ℓ-ideals I of A such that $I \cap B \subseteq J$; further, $\{a \in A \mid [a] \cap B \subseteq J\}$ is an ℓ-ideal by the flatness of $\mathfrak{L}\varepsilon$, and hence this is clearly it. □

Corollary. *For any f-ring A, the identical embedding $A^* \to A$ induces a homeomorphism* $\operatorname{Max} A^* \to \operatorname{Max} A$ *taking P to $\{a \in A \mid [a] \cap A^* \subseteq P\}$.*

Remark. Henriksen and Johnson [16] proved that $\operatorname{Max} A^* \cong \operatorname{Max} A$ under the additional assumption that A is archimedean and an $\mathbb{R}$-algebra. They used the representation of such f-rings by extended real-valued continuous functions on compact Hausdorff spaces, an approach radically different from the present proof of the general result, apart from its dependence on the PIT. For even more special f-rings this isomorphism goes back to [11]; the original result of this kind, concerning the case $A = C(X)$, the ring of continuous real-valued functions on a Tychonoff space X, was proved by Gelfand and Kolmogoroff [14].

3.4. It may be of interest to see how the more familiar form of the isomorphism of Corollary 3.3 is obtained in the case $A = C(X)$ for a Tychonoff space X with a universal compactification $Y \supseteq X$ ([15, 7.3]).

To begin with, recall there is a frame homomorphism

$$\sigma_X : \mathfrak{M}(C^*(X)) \to \mathfrak{O}X, \quad J \mapsto \bigcup\{Coz(\gamma) \mid \gamma \in J\}$$

$$Coz(\gamma) = \gamma^{-1}[\mathbb{R} - \{0\}]$$

which is natural in X, dense onto, and an isomorphism iff $\mathfrak{O}X$ (or equivalently: X) is compact [6], providing the coreflection map from compact completely regular frames and linking the open sets of X with the saturated ℓ-ideals of $C^*(X)$. We note in passing that the latter are, in fact, exactly the ring ideals of $C^*(X)$ closed in the usual norm topology [5]. As a further link between the open sets of X and saturated ℓ-ideals of functions we have:

Lemma. *For any* $U \in \mathfrak{O}X$, $K_U = \{\varphi \in C(X) \mid Coz(\varphi) \subseteq U\}$ *is a saturated* ℓ*-ideal of* $C(X)$; *in particular*, $\langle\gamma\rangle = K_{Coz(\gamma)}$ *for any* $\gamma \in C(X)$.

Proof. That K_U is an ℓ-ideal follows easily from the properties of cozero sets. To see it is saturated, one notes first, for any K_U-small ℓ-ideal J of $C(X)$ and $\gamma \geqslant 0$ in J, that $(\gamma - \mathbf{p})^+ \in K_U$ for any rational $p > 0$ (bold face indicating constant functions) because $\gamma + (\mathbf{p} - \gamma)^+ \geqslant \mathbf{p}$ and $(\mathbf{p} - \gamma)^+ \wedge (\gamma - \mathbf{p})^+ = 0$. Consequently,

$$\{x \in X \mid p < \gamma(x)\} = Coz(\gamma - \mathbf{p})^+ \subseteq U$$

for all $p > 0$, hence $Coz(\gamma) \subseteq U$ so that $\gamma \in K_U$, showing $J \subseteq K_U$.

In particular, it follows that $\langle\gamma\rangle \subseteq K_{\mathrm{Coz}(\gamma)}$, and the reverse inclusion is obtained by proving that $K_{\mathrm{Coz}(\gamma)}$ is $\langle\gamma\rangle$-small: if $K_{\mathrm{Coz}(\gamma)} + H = [\mathbf{1}]$ in $\mathfrak{L}C(X)$ then $\mathbf{1} \leqslant \alpha + \beta$ for some $\alpha \geqslant 0$ in $K_{\mathrm{Coz}(\gamma)}$ and $\beta \geqslant 0$ in H, hence $X = Coz(\alpha) \cup Coz(\beta)$ and therefore also $X = Coz(|\gamma| + \beta)$; this shows $|\gamma| + \beta$ is invertible and consequently $\langle\gamma\rangle + H = [\mathbf{1}]$, as desired. $\square$

Now we can readily derive the following familiar assertion (where $Z(\cdot)$ stands for zero set) from Corollary 3.3.

For any Tychonoff space X *which has a universal Tychonoff compactification* $Y \supseteq X$, *there is a homeomorphism* $Y \to \mathrm{Max}\, C(X)$ *taking* $y \in Y$ *to* $\{\gamma \in C(X) \mid y \in cl_Y Z(\gamma)\}$.

First, the lemma immediately implies that

$$(\sigma_X)_*(Coz(\gamma)) = K_{\mathrm{Coz}(\gamma)} \cap C^*(X) = \langle\gamma\rangle \cap C^*(X) \quad (\gamma \in C(X))$$

for the above frame homomorphism $\sigma_X : \mathfrak{M}C^*(X) \to \mathfrak{O}X$. Next, the hypothesis on Y determines an isomorphism $\ell : \mathfrak{O}Y \to \mathfrak{M}C^*(X)$ such that $\sigma_X \ell = \varrho$ for $\varrho : \mathfrak{O}Y \to \mathfrak{O}X$ taking each $U \in \mathfrak{O}Y$ to $U \cap X$. Further, for any $J \in \mathfrak{M}C^*(X)$ and $\gamma \in C(X)$

$$\langle\gamma\rangle \cap C^*(X) \subseteq J \quad \text{iff} \quad [\gamma] \cap C^*(X) \subseteq J$$

by the proof of part (1) of Supplement 2.3 which shows that $\langle\gamma\rangle \cap C^*(X)$ is $[\gamma] \cap C^*(X)$ – small. Finally, for any $\gamma \in C(X)$, $P \in \mathrm{Max}\, C^*(X)$, and $y \in Y$ such that $P = \ell(Y - \{y\})$,

$$[\gamma] \cap C^*(X) \subseteq P \quad \text{iff} \quad \langle\gamma\rangle \cap C^*(X) \subseteq P \quad \text{iff}$$

$$(\sigma_X)_*(Coz(\gamma)) \subseteq P \quad \text{iff} \quad \varrho_*(Coz(\gamma)) \subseteq \ell^{-1}(P) = Y - \{y\} \text{ iff}$$

$$Y - cl_Y(X - Coz(\gamma)) \subseteq Y - \{y\} \quad \text{iff} \quad y \in cl_Y(Z(\gamma)).$$

Hence the desired result by Corollary 3.3.

This proves Theorem 7.3 of [15]. Note that the present argument does not assume the *general existence* of the Stone-Čech compactification.

Remark. Without the intervention of Y the homeomorphism of Corollary 3.3 is described as taking $P \in \operatorname{Max} C^*(X)$ to

$$\{\gamma \in C(X) \mid (\sigma_X)_*(Coz(\gamma)) \subseteq P\}.$$

In this form, the same result holds for the rings of real-valued continuous functions on an arbitrary completely regular frame. For the functions as such see [20], and for a treatment of these rings [6].

3.5. There are results perfectly parallel to those of 3.4 in the case of f-rings $C(X,\mathbb{Z})$ of integer-valued continuous functions on zero-dimensional spaces X. In particular, the isomorphism $\mathfrak{M}C^*(X,\mathbb{Z}) \to \mathfrak{M}C(X,\mathbb{Z})$ of Corollary 3.3 gives rise to the following:

For any zero-dimensional space X which has a universal zero-dimensional compactification $Z \supseteq X$, there is a homeomorphism $Z \to \operatorname{Max} C(X,\mathbb{Z})$ taking $z \in Z$ to $\{\gamma \in C(X,\mathbb{Z}) \mid z \in cl_Z Z(\gamma)\}$.

In addition, there is an analogue of Remark 3.4 for the f-rings of integer-valued continuous functions on arbitrary zero-dimensional frames L whose elements are the frame homomorphisms $\mathfrak{P}\mathbb{Z} \to L$ and whose operations are appropriately induced by the operations of $\mathbb{Z}$ as f-ring.

4. Gelfand rings

4.1. We begin with some general observations about arbitrary rings, always understood to have a unit but specifically not assumed to be commutative. Further, ring homomorphisms are taken to be unit preserving.

For any ring A, an *ideal* of A is meant to be a two-sided ideal, and $Id\,A$ will be the complete lattice of all these which forms a special type of quantale [25] with respect to the operation $\bigvee$ and the usual multiplication

$$IJ = \left\{ \sum_{i=1}^{n} a_i b_i \mid a_i \in I,\ b_i \in J,\ n = 1, 2, \dots \right\}.$$

For any $a \in A$, $[a]$ will be the principal ideal generated by a. An ideal $J \subseteq A$ is called a *Baer radical ideal* if $I^n \subseteq J$ implies $I \subseteq J$ for any $I \in Id\,A$ and natural n. Clearly, these J form a closure system in $Id\,A$, here called $BRId\,A$, but in fact this is a *frame*; more specifically, it is the frame reflection of the quantale $Id\,A$, that is, the map $Id\,A \to BRId\,A$ by the corresponding closure operator is the universal $\bigvee$-homomorphism to frames taking the products IJ to meets [23].

Further, an ideal $J \subseteq A$ is called a *Brown–McCoy radical ideal* if $J = b(J)$ for the operator b on $Id\,A$ defined by

$$b(J) = \bigvee \{I \in Id\,A \mid J + [1+a] = [1] \text{ for all } a \in I\}.$$

It turns out that b is a codense closure operator such that

$$b(I) \cap b(J) = b(I \cap J) = b(IJ)$$

and the corresponding closure system $BId\,A$ is a frame, characterized by a certain extremality condition. Moreover, the ideals I in the definition of $b(J)$ are exactly the J-small ideals, extending the terminology of 1.7 to $Id\,A$, and hence b is none other than the generalization of the saturation nucleus of a compact frame [7].

Note that the maximal spectrum $\mathrm{Max}\,A$ of a ring A, that is, its space of maximal ideals, is a subspace of the spectrum of the frame $BId\,A$, in general properly smaller.

4.2. Although $BRId\,A$ is derived from $Id\,A$ by a functorial process it does not seem to depend functorially on A itself except for commutative A in which case $A \mapsto Id\,A$ is actually a functor from rings to certain quantales. There is, however, a reasonable way out of this by passing to a suitable quotient of $BRId\,A$, reasonable because it does not affect the commutative case.

To describe this, recall that a *symmetric* ideal J of a ring A is one such that $abc \in J$ implies $acb \in J$ for any $a, b, c \in A$. Concerning these one has the following observations [21]:

4.2.1. For any symmetric ideal J, if $a_1, a_2 \ldots a_n \in J$ then also $a_{\sigma(1)}a_{\sigma(2)}\ldots a_{\sigma(n)} \in J$ for any permutation σ of $\{1, 2, \ldots, n\}$.

4.2.2. A symmetric ideal J is a Baer radical ideal iff $a^n \in J$ implies $a \in J$ for any $a \in A$ and natural n.

4.2.3. Any directed union of symmetric Baer radical ideals is again such an ideal.

We shall call the symmetric Baer radical ideals of a ring A the *radical ideals* of A and let $RId\,A$ be the corresponding closure system in $BRId\,A$.

Note that $RId\,A$ may well be trivial, as is the case for any simple ring A with elements a and b such that $ab = 0$ but $ba \neq 0$: here, the only symmetric ideal is A. Evidently this applies to matrix rings over fields.

The properties of $RId\,A$ required here are that

(1) *for any ring A, $RId\,A$ is a coherent frame*

and

(2) *the correspondence $A \mapsto RId\,A$ is functorial such that the frame homomorphism $RId\,A \to RId\,B$ determined by any ring homomorphism $\varphi : A \to B$ takes each $J \in RId\,A$ to the radical ideal of B generated by its image $\varphi[J]$.*

For the proof of these we refer to the appendix.

4.3. We now turn to the specific topic of this section.

A ring A is called a *Gelfand ring* if, for any distinct maximal left ideals K and L of A there exist elements $a \notin K$ and $b \notin L$ such that $aAb = \{0\}$ [24].

Note that $aAb = \{0\}$ is the same as saying $[a][b] = [0]$.

For typical examples of Gelfand rings see [24]. Further, we need from [24] that every maximal ideal of a Gelfand ring is a maximal left ideal and, as an easy consequence, symmetric.

We note that the proof of this requires the Axiom of Choice, as does the following result which is crucial for our purposes.

*****Lemma.** *For any Gelfand ring A, the frame $RId\,A$ is normal.*

Proof. By the properties just quoted, the maximal elements of $RId\,A$ are maximal left ideals. Hence, by the definition of Gelfand rings and by the proof of A.1 in the appendix, $RId\,A$ is a compact frame of the kind considered in 2.7 and hence normal. □

***4.4. Lemma.** *For any Gelfand ring A, $S(RId\,A) = BId\,A$.*

Proof. Assuming the Axiom of Choice, the closure operator $\ell = k\varrho$ on $Id\,A$ associated with $RId\,A$ (A.1) is codense: for any proper ideal J of A, if $P \supseteq J$ is a maximal ideal then $\ell(P) = P$ since P is symmetric, hence $\ell(J) \subseteq P$ so that $\ell(J)$ is still proper. It follows that ℓ is a multiplicative nucleus on $Id\,A$ in the sense of [7] and by Section 3 of the latter this implies that $BId\,A = S(RId\,A)$. □

Remark 1. Note that this lemma expresses a very special property of Gelfand rings: in general, $RId\,A$ and $S(RId\,A)$ may be trivial whereas $BId\,A$ never is. On the other hand, though, $S(RId\,A) = BId\,A$ for *any commutative* A.

As an immediate consequence of the lemma and of Lemma 4.3 we now have by 2.3 and 2.4 (where $\langle\cdot\rangle$ indicates principal radical ideals):

Proposition. *For any Gelfand ring A, the correspondences $A \mapsto BId\,A$ and $A \mapsto \mathrm{Max}\,A$ are functorial such that*

$$BId\,\varphi(J) = s_{RId\,B}\left(\bigvee\{\langle\varphi(a)\rangle \mid \langle a\rangle \prec J \text{ in } RId\,A\}\right)$$

and

$$\mathrm{Max}\,\varphi(P) = s_{RId\,A}(\varphi^{-1}[P])$$

for any homomorphism $\varphi : A \to B$. Moreover, $BId\,A$ is a compact regular frame and $\mathrm{Max}\,A$ *a compact Hausdorff space.*

Remark 2. The functoriality of $A \mapsto \mathrm{Max}\,A$ is the result of Mulvey [24] referred to earlier, incidentally with exactly the same specification of $\mathrm{Max}\,\varphi$: [24] describes this as taking each $P \in \mathrm{Max}\,B$ to the unique maximal ideal above $\varphi^{-1}[P]$.

Remark 3. In general the correspondence $A \mapsto \mathrm{Max}\,A$ is *not* functorial as is shown by Banaschewski and Vermeulen [10] with the aid of suitable localizations of the polynomial rings in one and two indeterminates over $\mathbb{F}_2$ or, alternatively, in two and three indeterminates over an arbitrary field.

4.5. We conclude with some comments on the extent to which the above considerations depend on the Axiom of Choice. Crucially, the definition of a Gelfand ring is such that any ring *without* maximal left ideals is vacuously Gelfand. Now, as is well-known, any failure of the Axiom of Choice produces a commutative ring with unit which has no maximal ideals [18,3] but in fact it determines, more specifically, such a ring for which $RId\,A$ is *not normal* from the non-normal coherent frame without maximal elements mentioned in 2.7, by [4]. As a result we then have that *the Axiom of Choice holds iff* $RId\,A$ *is normal for every Gelfand ring* A.

In a sense this seems to indicate that the original definition of a Gelfand ring was somewhat unfortunate. On the other hand, simply defining a ring A to be Gelfand if $RId\,A$ is normal would seem unappealing for other reasons. However, in the commutative case at least, there is a suggestive way out of the dilemma: it turns out that the normality of $RId\,A$ is equivalent to the condition that, whenever $a+b=1$ in A, there exist $r,s\in A$ such that,

$$(1+ar)(1+bs)=0$$

(which is even first order), and one might therefore adopt the latter as the definition of Gelfand rings. That this already follows from the original condition in the presence of the Axiom of Choice could then be perceived as a minor curiosity.

With this revised notion, incidentally, it becomes meaningful to ask about the force of the condition that $\mathrm{Max}\,A\neq\emptyset$ for all commutative Gelfand rings A. Since this is ultimately the same as the condition that every coherent normal frame have maximal elements, and hence by 2.1 that every compact regular frame have maximal elements, it follows that it is equivalent to the PIT—a nice observation given that $\mathrm{Max}\,A\neq\emptyset$ for *arbitrary* A is equivalent to the Axiom of Choice.

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Appendix

Here, we provide the proofs for the two results concerning the lattice $RId\,A$ of radical ideals of a ring A quoted in 4.2.

A.1. *RId A is a coherent frame.*

Proof. To see that it is a frame consider the operator k_0 defined on the frame $BRId\,A$ (4.1) by

$$k_0(J) = \varrho(J + T_J), \quad T_J = \bigvee\{[acb] \mid abc \in J\} \text{ in } Id\,A$$

where ϱ is the closure operator on $Id\,A$ associated with $BRId\,A$. Clearly, $RId\,A = \mathrm{Fix}(k_0)$; hence if we show that k_0 is a prenucleus, that is,

$$J \subseteq k_0(J), \quad I \subseteq J \text{ implies } k_0(I) \subseteq k_0(J), \quad I \cap k_0(J) \subseteq k_0(I \cap J),$$

it will follow that the associated closure operator k is a nucleus, making $RId\,A$ a frame [2].

The first two conditions being obvious we concentrate on the third. For this, recall the basic property of ϱ that

$$\varrho(G) \cap \varrho(H) = \varrho(GH)$$

for any $G, H \in Id\,A$ [23]. Now, for any $I, J \in BRId\,A$,

$$I \cap k_0(J) = \varrho(I) \cap \varrho(J + T_J) = \varrho(IJ + IT_J) \subseteq \varrho(I \cap J + T_{I \cap J}) = k_0(I \cap J),$$

the third step because $IJ \subseteq I \cap J$ trivially while

$$IT_J = \bigvee\{I[acb] \mid abc \in J\} \subseteq T_{I \cap J}$$

since $x \in I$ and $abc \in J$ implies $xabc \in I \cap J$ and hence $xacb \in T_{I \cap J}$.

Next, the compact elements of $RId\,A$ are exactly the finitely generated ones in view of 4.2.3; in particular, $RId\,A$ is compact. Moreover, putting $\langle c \rangle = k\varrho([c])$, we have for any $a, b \in A$

$$\langle a \rangle \cap \langle b \rangle = k\varrho([a]) \cap k\varrho([b]) = k(\varrho([a]) \cap \varrho([b]))$$

$$= k\varrho([a][b]) \subseteq \langle ab \rangle,$$

the last step since $[a][b] \subseteq \langle ab \rangle$. It follows that $\langle a \rangle \cap \langle b \rangle = \langle ab \rangle$, and this readily ensures that the meet of any two compact elements of $RId\,A$ is compact. □

A.2. *The correspondence $A \mapsto RId\,A$ is functorial such that the frame homomorphism $RId\,A \to RId\,B$ determined by any ring homomorphism $\varphi : A \to B$ takes each $J \in RId\,A$ to $\bigvee\{\langle \varphi(a) \rangle \mid a \in J\}$ in $RId\,B$.*

Proof. We first have to show that the map described here is a frame homomorphism. For this, if $\bar{\varphi}(J)$ is the ideal generated by $\varphi[J]$ in B and $\tilde{\varphi}(J) = \ell(\bar{\varphi}(J))$ for the composite $\ell = k\varrho$ on $Id\,B$ then $\tilde{\varphi}(J)$ is obviously the radical ideal indicated above. Note that $I \mapsto \varphi^{-1}[I]$ maps $RId\,B$ to $RId\,A$: it clearly takes symmetric ideals to symmetric ideals and consequently radical ideals to radical ideals by 4.2.2. On the other hand,

$$\tilde{\varphi}(J) \subseteq I \quad \text{iff} \quad \bar{\varphi}(J) \subseteq I \quad \text{iff} \quad J \subseteq \varphi^{-1}[I]$$

for any $J \in RId\,A$ and $I \in RId\,B$, showing that $\tilde{\varphi}$ has a right adjoint and hence preserves arbitrary joins. As it obviously preserves the unit it remains to check binary meet. For any $I, J \in RId\,A$,

$$\tilde{\varphi}(I) \cap \tilde{\varphi}(J) = \ell(\bar{\varphi}(I)) \cap \ell(\bar{\varphi}(J)) = \ell(\bar{\varphi}(I)\bar{\varphi}(J))$$

$$\subseteq \ell(\bar{\varphi}(IJ)) \subseteq \ell\bar{\varphi}(I \cap J) = \tilde{\varphi}(I \cap J),$$

the second step by the properties of k and ϱ and the third because $\varphi[I]\varphi[J] \subseteq \ell\bar{\varphi}(IJ)$ trivially and hence $\bar{\varphi}(I)\bar{\varphi}(J) \subseteq \ell\bar{\varphi}(IJ)$ by the symmetry of the latter. This proves the non-trivial part of the desired identity.

To obtain the required properties of the correspondence $\varphi \mapsto \tilde{\varphi}$, we begin by showing that the diagram

$$\begin{tikzcd} Id\,A \arrow[r, "\bar{\varphi}"] \arrow[d, "\ell_A"'] & Id\,B \arrow[d, "\ell_B"] \\ RId\,A \arrow[r, "\tilde{\varphi}"'] & RId\,B \end{tikzcd}$$

commutes for any ring homomorphism $\varphi : A \to B$. For any $J \in Id\,A$ and $I \in RId\,B$,

$$\tilde{\varphi}\ell_A(J) = \ell_B\bar{\varphi}(\ell_A(J)) \subseteq I \quad \text{iff} \quad \bar{\varphi}(\ell_A(J)) \subseteq I$$

$$\text{iff} \quad \ell_A(J) \subseteq \varphi^{-1}[I] \quad \text{iff} \quad J \subseteq \varphi^{-1}[I] \quad \text{iff} \quad \bar{\varphi}(J) \subseteq I$$

$$\text{iff} \quad \ell_B\bar{\varphi}(J) \subseteq I,$$

the crucial third step because $\varphi^{-1}[I] \in RId\,A$.

Now, for any further ring homomorphism $\psi : B \to C$, straightforward checking shows that $\overline{\psi\varphi} = \bar{\psi}\bar{\varphi}$, and using the above commuting square for both φ and ψ we obtain

$$(\psi\varphi)^{\sim}\ell_A = \ell_C\overline{\psi\varphi} = \ell_C\bar{\psi}\bar{\varphi} = \tilde{\psi}\ell_B\bar{\varphi} = \tilde{\psi}\tilde{\varphi}\ell_A.$$

It follows that $(\psi\varphi)^{\sim} = \tilde{\psi}\tilde{\varphi}$, and since $\tilde{\varphi} = id_{RId\,A}$ trivially for $\varphi = id_A$, this shows we have a functor, as claimed. $\square$

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